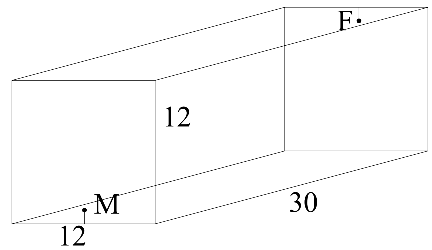
Two Ants

A rectangular room has dimensions 12×12×30 (feet). There is a vertical line running up the middle of each of the square walls. A male ant is on one of these lines, 1 foot from the floor, and a female ant is on the other, 1 foot from the ceiling. How far does the male have to crawl to get to the female? We will, of course, assume that the male follows the shortest route to get to the female, given that he has to stick to the floor, the walls or the ceiling. That is, although some male ants have wings and can fly, this one cannot.



This problem is found in Ball and Coxeter Mathematical Recreations & Essays *[University of Toronto Press, 12th ed. 1974. page 120] and they attribute it to H.E. Dudeney. A similar question appeared in print in the London Daily Mail in February of 1905.*

I throw the problem out to the class and await their results which I expect will be not long in coming. Indeed, I intend to use this problem as a warm-up and move quickly on to some juicier problem of the same type (e.g. Problem 3).

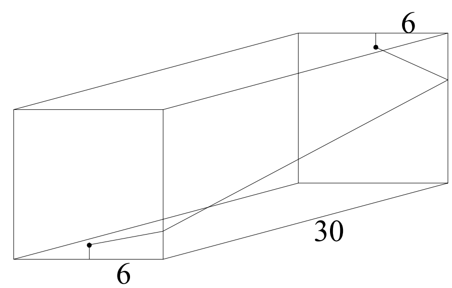
But I am astounded—the result that I expected from them does not come!

Solution. I wander among the student groups and find that they have all identified the obvious "floor-endwall" route which runs down the line to the floor, across the floor, and up the opposite line to the female: a total distance of 1+30+11 = 42 feet.

However, they know that I have a fondness for mathematical trickery, so they spend some time trying to find other shorter paths. But they do not succeed.

In fact, someone volunteers to "prove" that 42 is the shortest distance, and comes resolutely to the board. He argues as follows, using the 3-dimensional picture as an aid.

An alternative to the "floor-endwall" route would be to use one of the side walls. Now to get over to the side wall requires at least 6 feet, and then to get to the other side of the room requires at least 30 feet, and then at least 6 feet must again be used to get back to the vertical line. That's 42 feet right there, and we haven't even made allowance for the change in height.



An argument that 42 is the minimum distance.

Hmm. An interesting argument. What about the ceiling—is that another alternative?

Well, if you're going to go over to the side wall, you'd never want to use the ceiling, as that would take you too high, and you'd just have to come back down.

Seems quite convincing. I take a vote: *how many think that the minimum distance is 42*? Every hand goes up—it's unanimous!

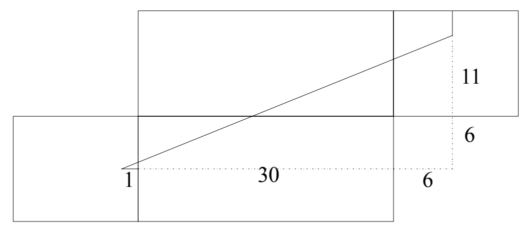
I stand at the front staring at the class, and they stare back at me expectantly. The argument seems fine—why am I not saying anything?

In fact, I suddenly don't know what to do. I did not expect this. I always want the new ideas to come from the class. There are 50 students in this session, and at the very least I expect some doubting Thomas or Thomasina to come forward with reservations, no matter how vague. But there is not a hint of revolt. How can this have happened? What am I to do?

In fact, I explode: ***Well you're wrong—you're simply wrong!*** *I am sufficiently loud and exasperated to get a few students peeking in from the hall to see what the excitement might be about. I invite them in but they see that it's a math class and they scurry away.*

Fortunately I have brought squared paper, scissors and tape. "Make boxes," I decree.

Most students embark with enthusiasm on this "cut and paste" activity, but a few continue to stare in puzzlement at their diagrams. As I wander around the room, I notice that without fail all of their pictures are 3-dimensional! Interesting, and not what I had expected. Perhaps the boxes they are making will allow them to see how to make use of 2-dimensional representations.

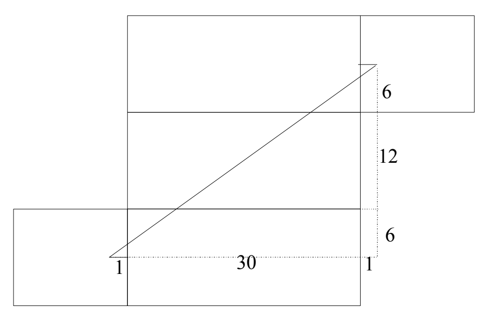


End-floor-sidewall-end: a route with
$D = \sqrt{37^2 + 17^2} = \sqrt{1658} \approx 40.72$

Indeed, quite soon there are 2-D diagrams sprouting all through the classroom. And in no time, a floor-sidewall solution appears on the board with a distance of 40.72 feet.

This is very nice—by opening out the box, the minimum path can be represented as a straight line. The guys who had "proved" that 42 was the minimum had to study this picture real carefully.

Now we have a nice technique for generating minimum paths. Can we use it to find other even shorter paths, or is 40.72 the best we can do? I try to take a vote, but there are few "takers." Most students have their noses buried determinedly in their papers.



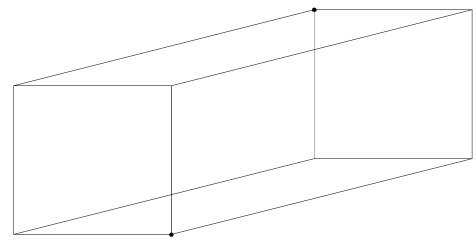
A route with
$D = \sqrt{32^2 + 24^2} = 8\sqrt{4^2 + 3^2} = 8 \cdot 5 = 40$

Before long another type of path appears on the board with a distance of exactly 40 feet. This one uses the ceiling as well as the side wall. The answer is a whole integer (and a nice one at that) so surely this must be the correct solution. Is it?

In fact it is. For any particular way of opening the box up, we get the shortest path by drawing a straight line, and so all we have to do is to make sure that all possible (or I should say all "reasonable") openings have been diagrammed.

Problems

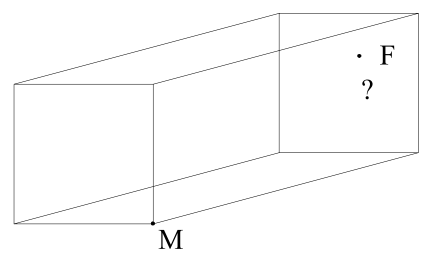
1.(a) A rectangular room has dimensions 1×1×2. An ant is at one corner of the room and must reach the diametrically opposite corner by crawling along the walls, floor or ceiling of the room. What is his shortest path?



(b) Same as (a) except the room has dimensions 1×1×*k* where *k* can be any positive number. Find the shortest path for each possible value of *k*.

2. A rectangular room has dimensions 1×1×2. Two ants, who are not on very good terms, have positioned themselves so they are as far apart as possible, where distance, of course, is measured as "crawling distance" along floor, walls or ceiling. Where are they and how far apart are they?

3. A rectangular room has dimensions 12×12×24. That is, the floor and ceiling and both the side walls are 12×24 and the two end walls are 12×12. In the room there are two ants, a male and a female. The male ant is on the floor at one of the corners.

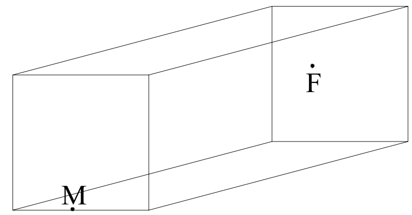


Now the female has positioned herself to be as far as possible from the male. That is, she has located herself at a point so that the male will take the longest possible time to get to her, given that he has to crawl along the walls, floor or ceiling of the room and will (of course) choose his path so that he gets to the female in the shortest possible time.

The question is: where is the female?

Well there's an obvious answer—the diametrically opposite corner. That's certainly the point which is farthest from the ant "as the crow flies." But an ant is not a crow .

4. A rectangular room has dimensions 2×2×4 and is oriented so that the two 2×2 walls are facing north and south. A male ant is at the base of the south wall, halfway between the east and west walls. A female ant is somewhere on the north wall, and the male wants to crawl to her following the shortest possible path. Now the type of route he will choose depends on exactly where the female is.



Draw a square representing the north wall, and identify and label different regions according to the path type that would be used to get to points in that region. For example, use F if only the floor is used, FW if the floor and the west wall are used, FEC if the floor, east wall and ceiling are used, etc.

As a hint, to save you a bit of work, I will tell you that the south wall would never be used. You could verify this yourself by examining a lot of cases, but the calculations are tedious.

This is an excellent problem, but it is not to be rushed. The student should be prepared (and allowed!) to spend a considerable amount of time on it. It is not overly difficult in the sense of requiring ingenuity and insight, but there's lots to be done: playing and checking things out, organizing the results, deciding how to present them, making good use of diagrams, and then writing it out. A good job is tremendously satisfying.

Well, you might reply, I can see that it's a nice problem, and no doubt it contains all the wonderful things you say it does, but my students simply don't have time to spend. There are so many techniques they have to learn and what does this problem have of those—Pythagoras and little else. How will they succeed in life if all they can do is solve problems like this?

Of course you already know my answer—I've said it 10 times already, and I'll say it 10 times more. They don't really need all those skills. What they really need is this.

Of course, there are some students who will need all the skills in the standard grade 12 text-book and more, for example, those who go on to study engineering physics. But how many such students do you teach? And such students will learn those skills through the years anyway, and those they miss they can pick up in a flash—otherwise they shouldn't be wrestling with engineering physics in the first place.