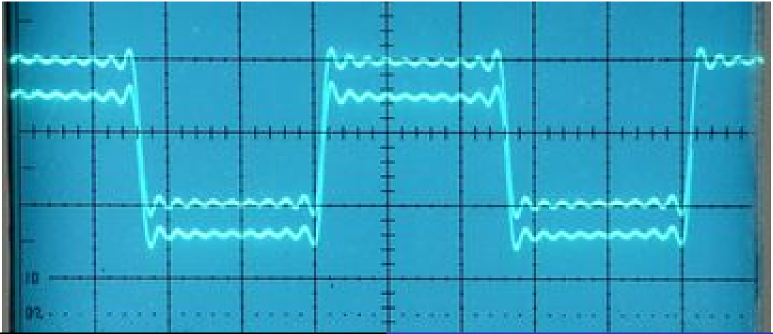
Physically Meaningful or Mathematically Pedantic?

Connor Behan

November 29, 2013



Something NOT physical

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The temperature of a metal spontaneously going to infinity!

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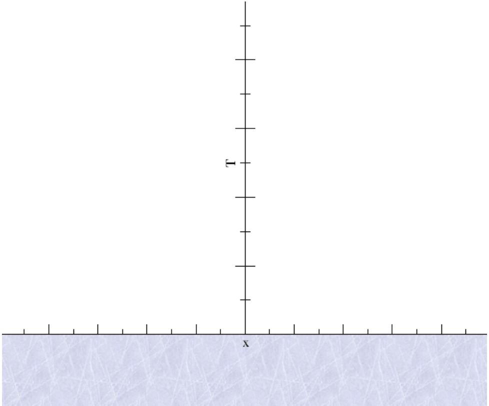

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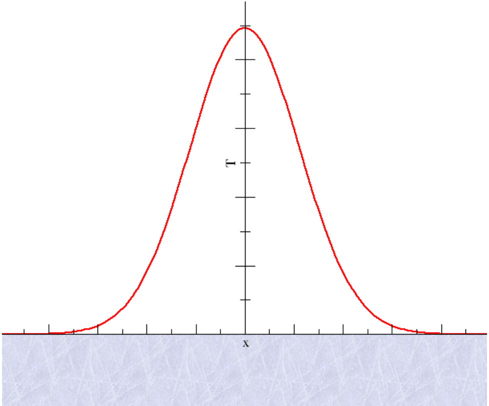

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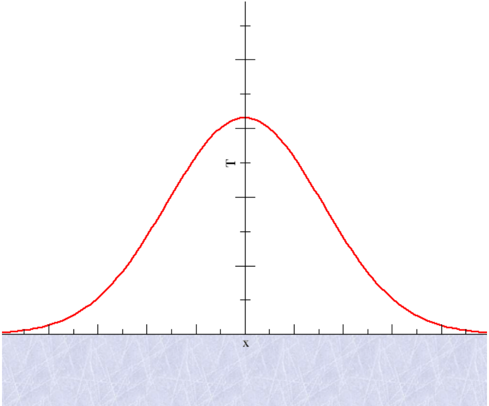

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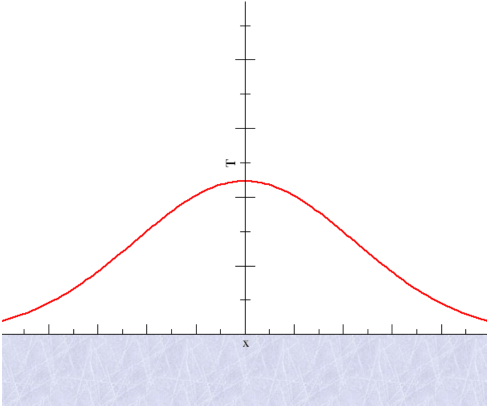

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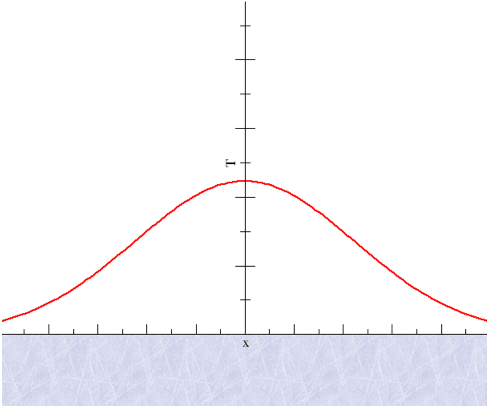

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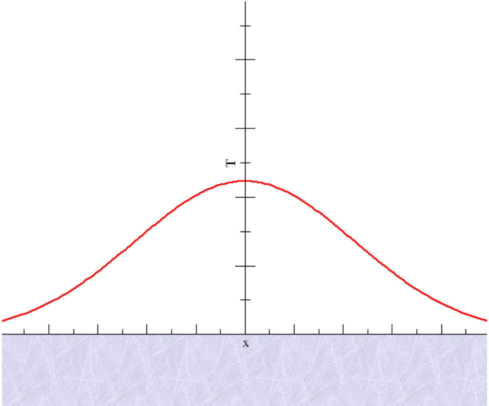

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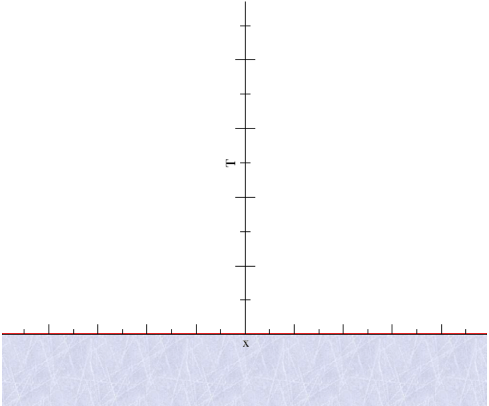

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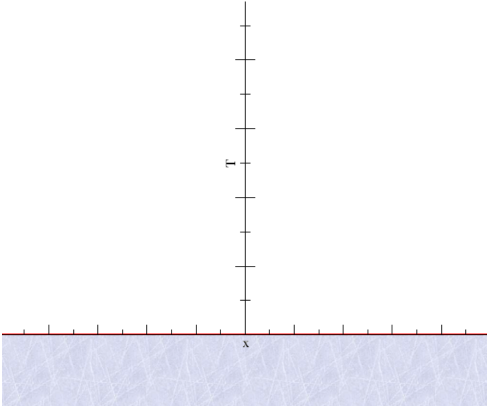

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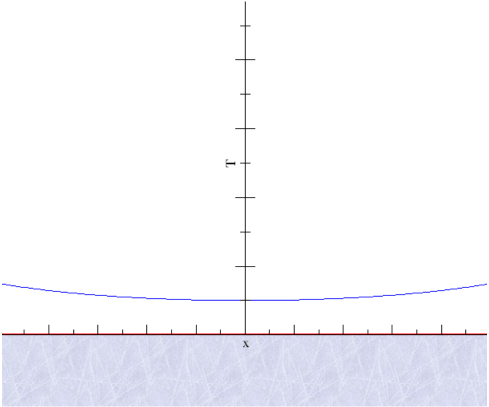

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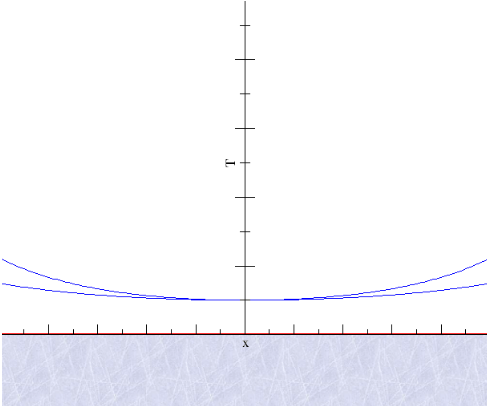

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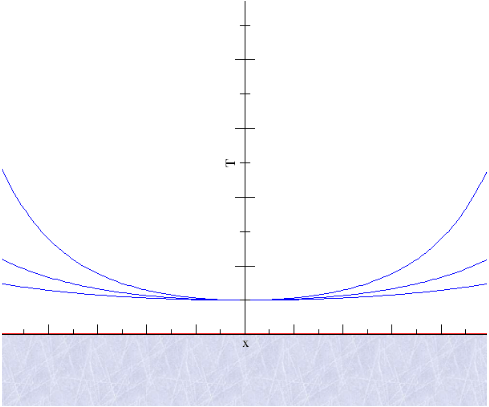

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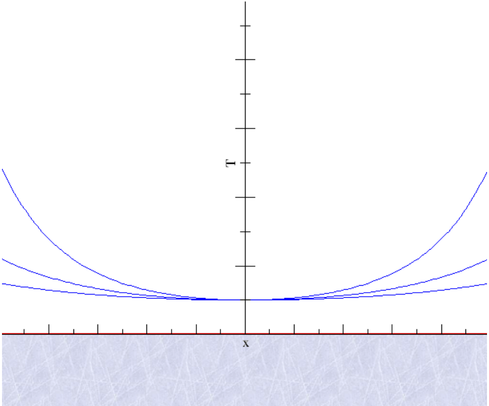

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Approximating a large system as infinite might not be safe.

Cubes of Salt

1cm

Cubes of Salt

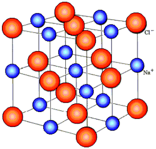
1cm 2cm 3cm

Cubes of Salt

1cm

2cm

3cm



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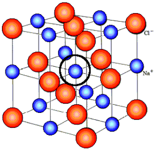

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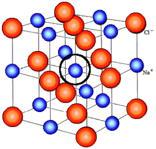

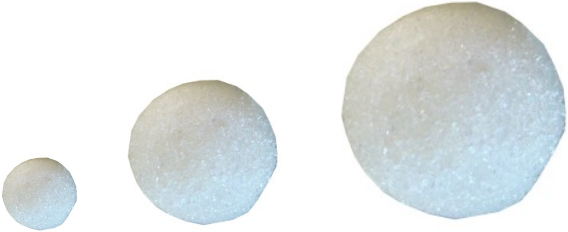

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$$+ \left(\frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19} \right) - \frac{1}{8} + \cdots = \infty$$

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The 6 nearest-neighbour Cl^- are one unit r_0 away.

$$V(0) \sim 6\frac{-e}{4\pi\epsilon_0 r_0}$$

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$$\begin{aligned} V(0) &= \frac{e}{4\pi\epsilon_0 r_0}M \\ &\equiv \frac{e}{4\pi\epsilon_0 r_0}\sum_{(i,j,k)\neq(0,0,0)}\frac{(-1)^{i+j+k}}{\sqrt{i^2+j^2+k^2}} \end{aligned}$$

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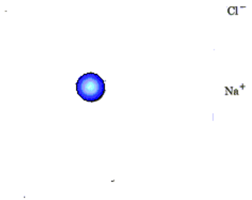
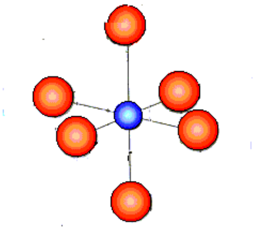
This sum is conditionally convergent!


Cl⁻
Na⁺

Cubes of Salt



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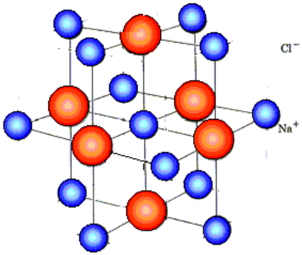

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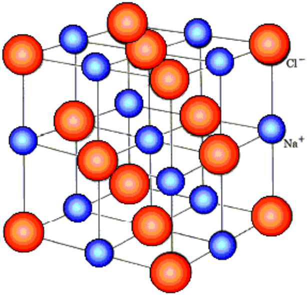

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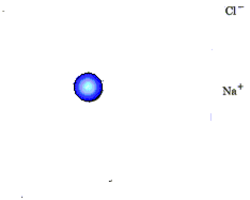

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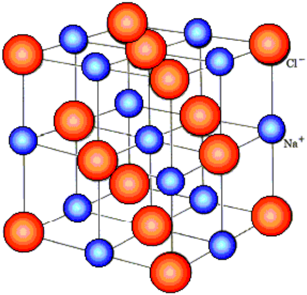

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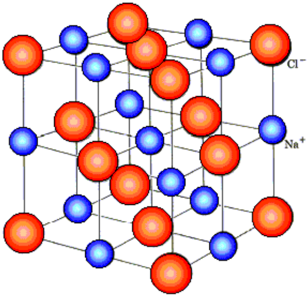

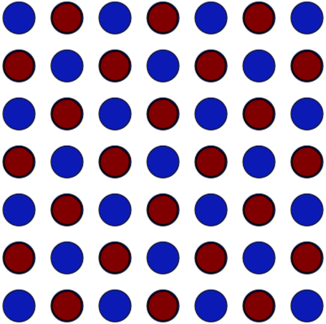

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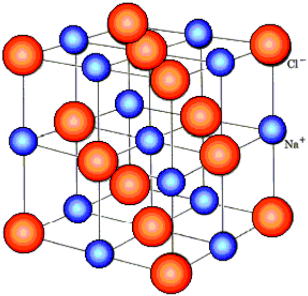

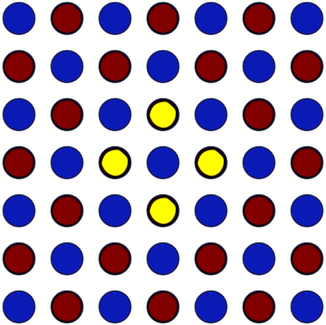

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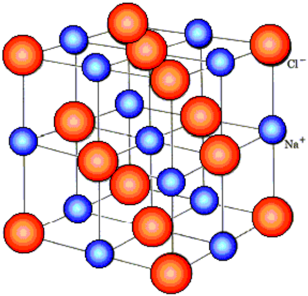

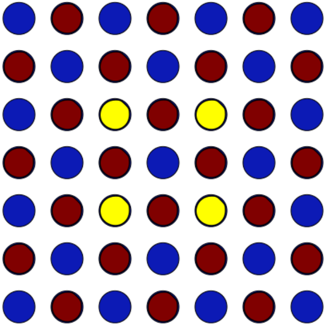

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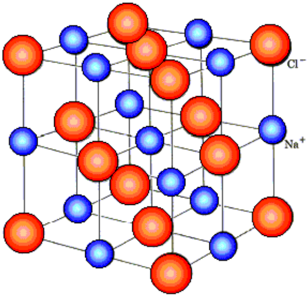

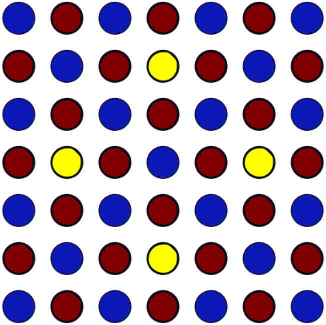

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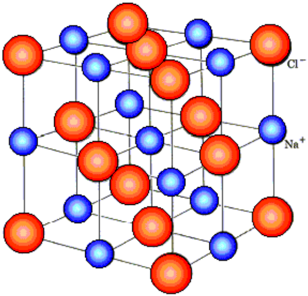

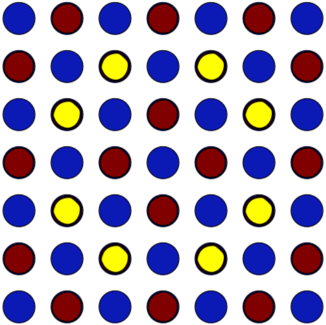

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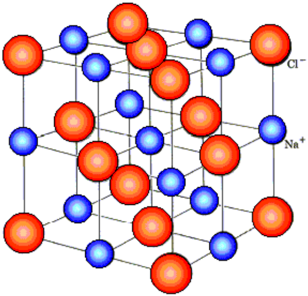

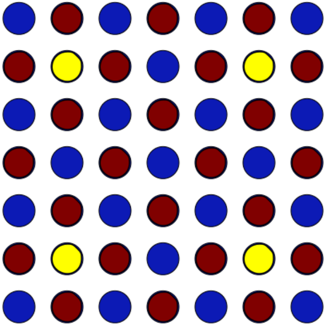

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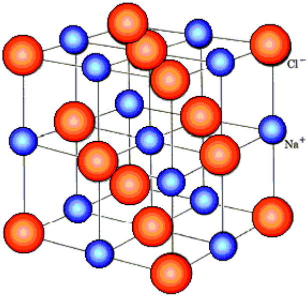

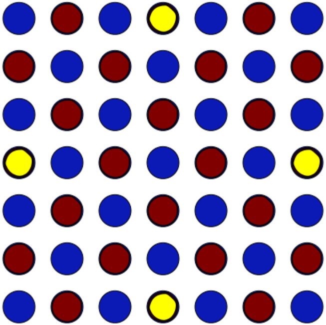

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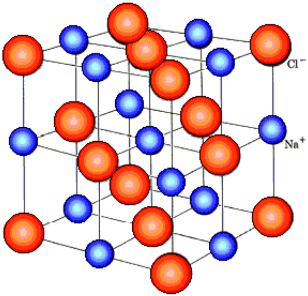

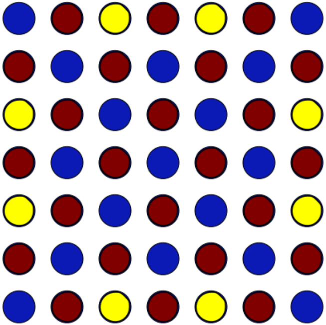

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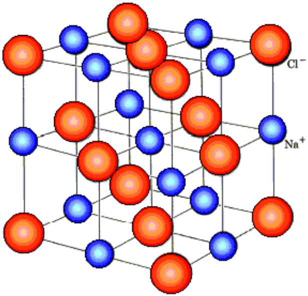

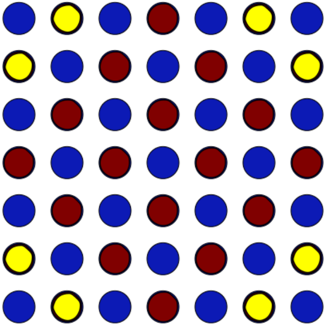

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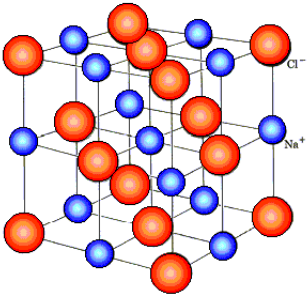

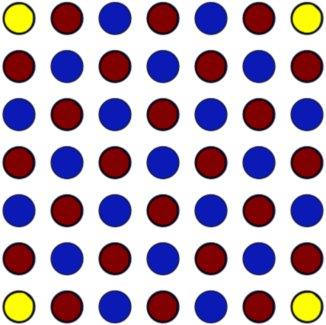

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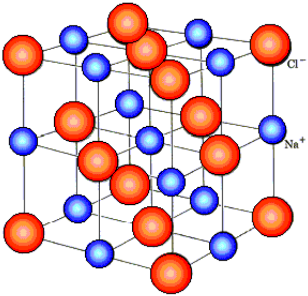

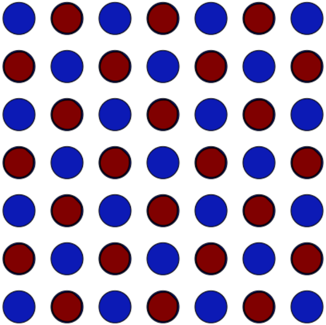

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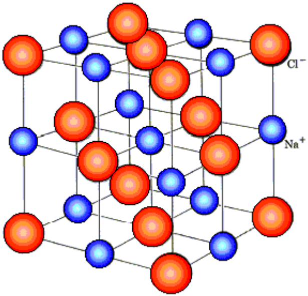

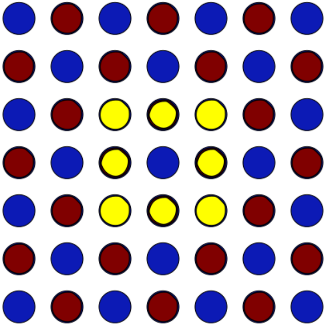

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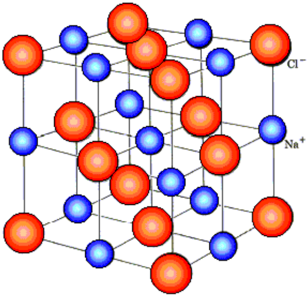

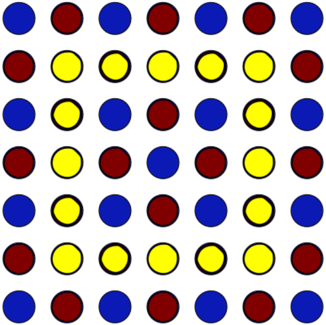

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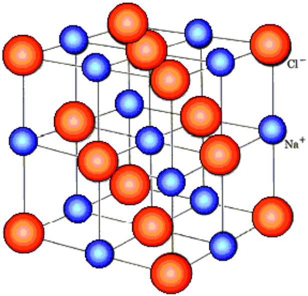

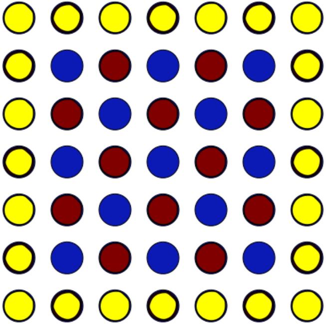

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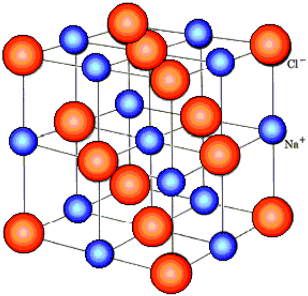

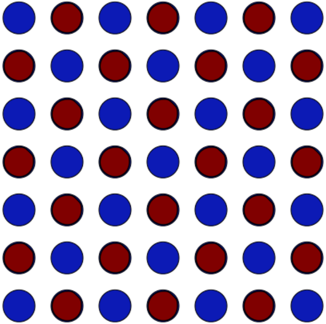

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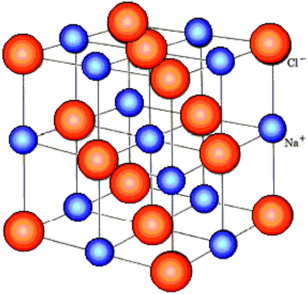
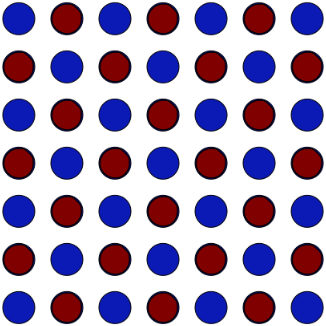
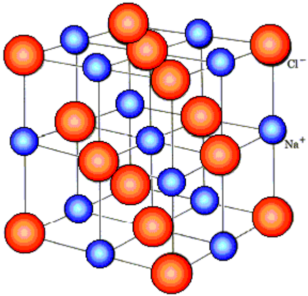
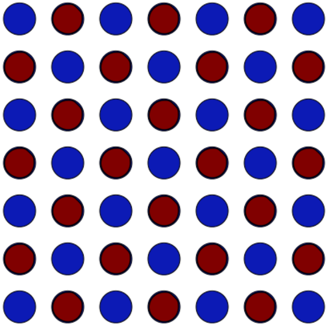




$M_{100} = -1.741820$ $M_{200} = -1.744685$ $M_{300} = -1.745643$

Cubes of Salt





$$M_{100} = -1.741820 \qquad M_{200} = -1.744685 \qquad M_{300} = -1.745643$$
$$M_{100} = 3.469987 \qquad M_{200} = -0.403582 \qquad M_{300} = 11.510973?$$

Hooke's Law

$$F(x) = k(x - x_0)$$

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Proof

$$\begin{aligned} U(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^n U}{\mathrm{d}x^n}(x_0)(x - x_0)^n \\ &\approx U(x_0) + U'(x_0)(x - x_0) + \frac{1}{2} U''(x_0)(x - x_0)^2 \\ &= U(x_0) + \frac{1}{2} U''(x_0)(x - x_0)^2 \end{aligned}$$

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What if U is not analytic?

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$$U(x) = \begin{cases} U_0 e^{-\frac{L^2}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

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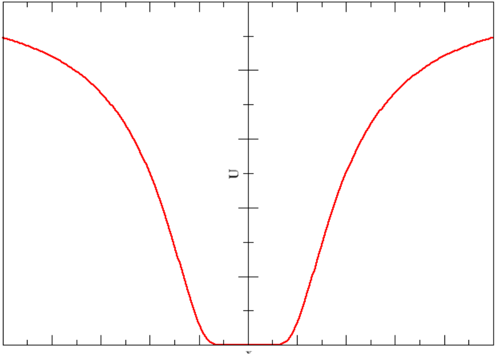

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Stone-Weierstrass Theorem: Polynomials are dense in $C([a,b];\mathbb{R})$.

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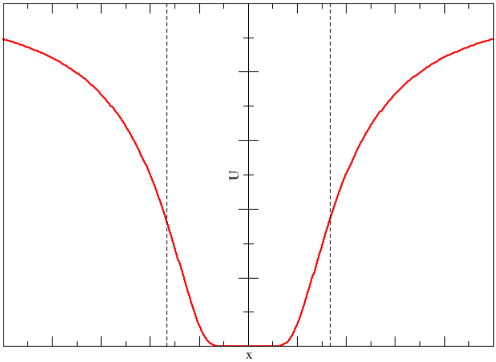

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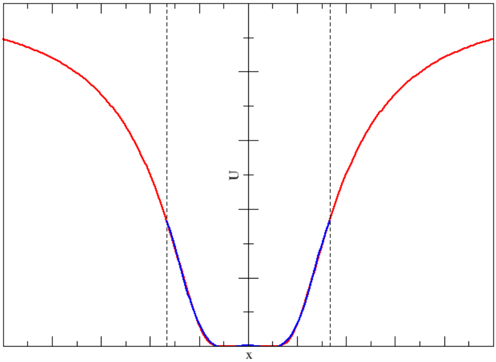

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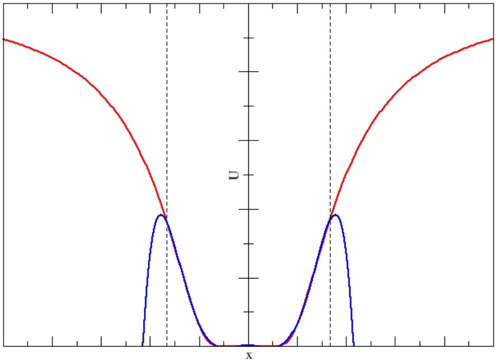

Hooke's Law

- $-0.00017 + 0.02511x^2 - 0.54440x^4 + 2.96252x^6$

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- $0.00224x^2 - 0.20889x^4 + 7.04392x^6 - 107.0x^8 + 756.5x^{10} - 2242.4x^{12} + 2414.0x^{14}$

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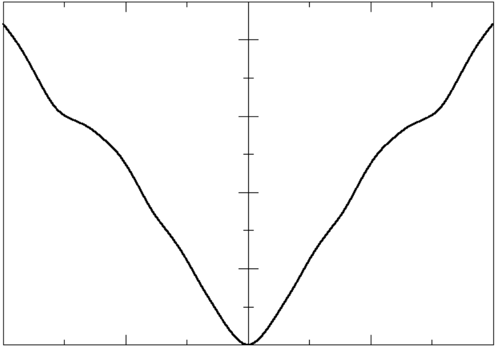
$$\sum_{k=1}^{\infty} e^{-2^{k/2}} 2^{kn} > e^{-2^{k_0/2}} 2^{k_0 n} = e^{-\sqrt{n}} n^n$$

if n is a power of 2.

Hooke's Law



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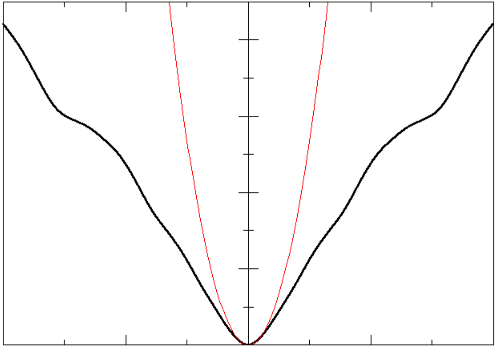

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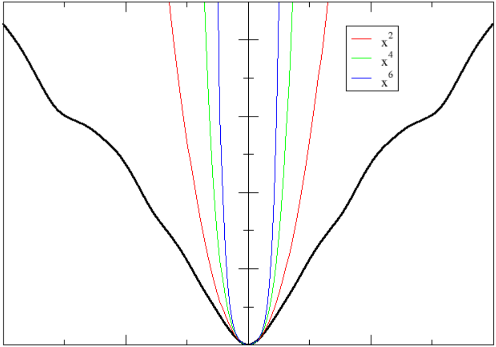

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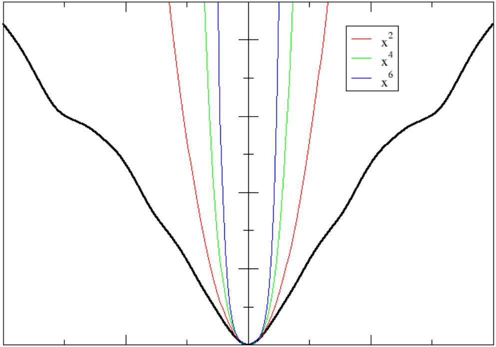

An asymptotic series for $f : \mathbb{R} \to \mathbb{R}$ is $\sum_{k=0}^{\infty} a_k x^k$ such that

$$\lim_{x \to 0} \frac{1}{x^n} \left[f(x) - \sum_{k=0}^{n} a_k x^k \right] = 0$$

A Little QFT

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1. Order matters.

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A Little QFT

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2. A divergent Taylor series can still be useful.
3. The infinite volume limit is not always unique.

A Little QFT

$$T \langle \psi_{\mathrm{out}} | \exp \left(-i \int_{-\infty}^{\infty} H_{\mathrm{int}} \mathrm{d}t \right) | \psi_{\mathrm{in}} \rangle$$

A Little QFT

$$\begin{aligned} & T \langle \psi_{\text{out}} | \exp\left(-i \int_{-\infty}^{\infty} H_{\text{int}} \mathrm{d}t \right) | \psi_{\text{in}} \rangle \\ \approx \quad & T \langle \psi_{\text{out}} | \left[1 - i \int_{-\infty}^{\infty} H_{\text{int}} \mathrm{d}t - \frac{1}{2} \left(\int_{-\infty}^{\infty} H_{\text{int}} \mathrm{d}t \right)^2 \right] | \psi_{\text{in}} \rangle \end{aligned}$$

A Little QFT

$$
\begin{aligned}
& T \langle 0 | a_k a_{k+q} \exp\left(-i \int_{-\infty}^{\infty} H_{\mathrm{int}} \mathrm{d}t \right) a_p^\dagger a_{p+q}^\dagger | 0 \rangle \\
\approx \quad & T \langle 0 | a_k a_{k+q} \left[1 - i \int_{-\infty}^{\infty} H_{\mathrm{int}} \mathrm{d}t - \frac{1}{2} \left(\int_{-\infty}^{\infty} H_{\mathrm{int}} \mathrm{d}t \right)^2 \right] a_p^\dagger a_{p+q}^\dagger | 0 \rangle
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\approx \quad & T\langle 0| a_k a_{k+q} \left[1 - \frac{1}{2}\left(\int_{-\infty}^{\infty} H_{\text{int}} \mathrm{d}t\right)^2 + \frac{1}{24}\left(\int_{-\infty}^{\infty} H_{\text{int}} \mathrm{d}t\right)^4\right] a_p^\dagger a_{p+c}
\end{aligned}
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\begin{aligned}
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= \; & c_1 \alpha + c_2 \alpha^2 + c_3 \alpha^3 + \dots \quad \alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}
\end{aligned}
$$

$$T\langle 0| a_k a_{k+q} \exp\left(-i\int_{-\infty}^{\infty} H_{\mathrm{int}}\,\mathrm{d}t\right) a_p^\dagger a_{p+q}^\dagger |0\rangle$$

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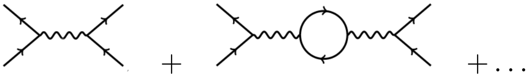
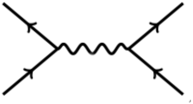


A Little QFT



$$\sum_{s,s'} \left| \mathrm{Tr}\, \bar{u}(p+q)\gamma_\mu u(p) \frac{-i\eta^{\mu\nu}}{q^2} \bar{u}(k+q)\gamma_\nu u(k) \right|^2$$

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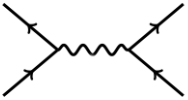

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$$= \frac{1}{q^2} [p \cdot k(p+q) \cdot (k+q) + p \cdot (k+q)k \cdot (p+q)$$

$$- m^2 p \cdot (p+q) - m^2 k \cdot (k+q) + 2m^4]$$

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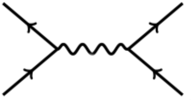

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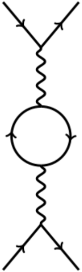
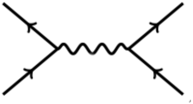


$$\sum_{s,s'} \left| \mathrm{Tr} \bar{u}(p+q)\gamma_\mu u(p) \frac{-i\Pi^{\mu\nu}(q)}{q^2} \bar{u}(k+q)\gamma_\nu u(k) \right|^2$$



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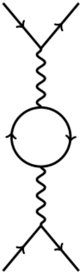
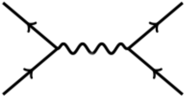


$$\sum_{s,s'} \left| \mathrm{Tr}\,\bar{u}(p+q)\gamma_\mu u(p) \frac{-i\Pi^{\mu\nu}(q)}{q^2} \bar{u}(k+q)\gamma_\nu \right.$$

$$\Pi^{\mu\nu}(q) = \int_{\mathbb{R}^4} \frac{\mathrm{Tr}\left[\gamma^\mu(\not{l}+m)\gamma^\nu(\not{l}+\not{q}+m)\right]}{(l^2-m^2)((l+q)^2-m^2)} \frac{dl}{(2\pi)^4}$$

A Little QFT



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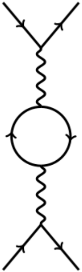

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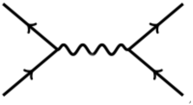
A Little QFT



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$$= \frac{1}{q^2} \left[p \cdot k (p+q) \cdot (k+q) + p \cdot (k+q) k \cdot (p+q) \right.$$

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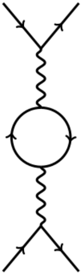

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$$\neq \lim_{d \to 4} \int_{\mathbb{R}^d} \frac{\mathrm{Tr}\left[\gamma^\mu (\not{l} + m)\gamma^\nu (\not{l} + \not{q} + m)\right]}{(l^2 - m^2)((l+q)^2 - m^2)} \frac{dl}{(2\pi)^d}$$

A Little QFT

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