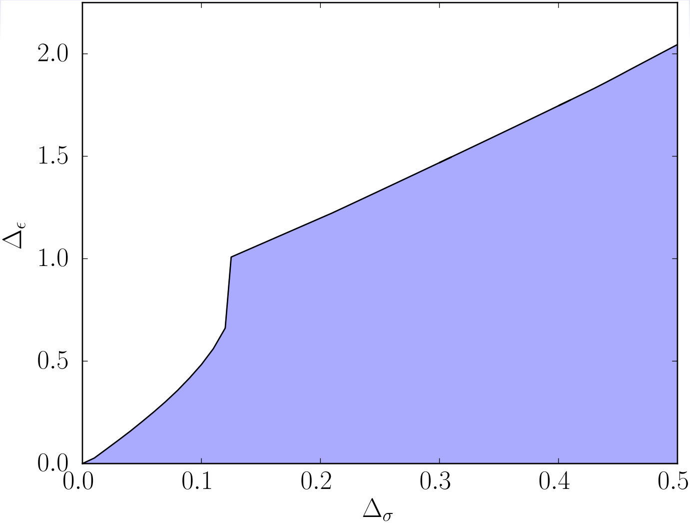
Unitarity versus Positivity

Connor Behan

May 27, 2021
Bootstat 2021

1712.06622, Upcoming?



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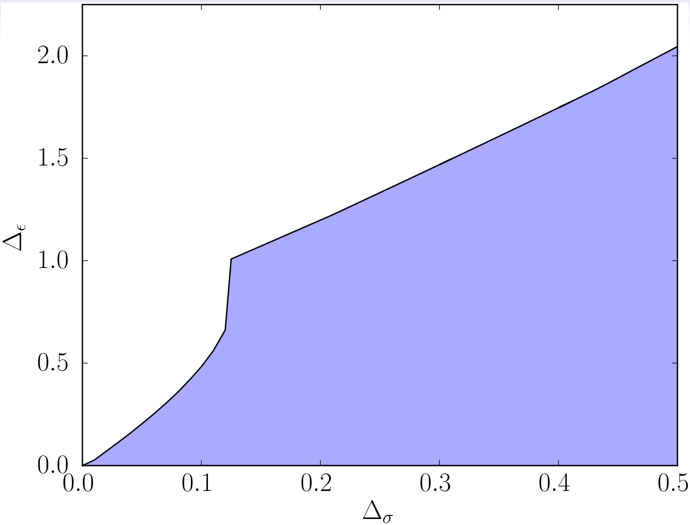
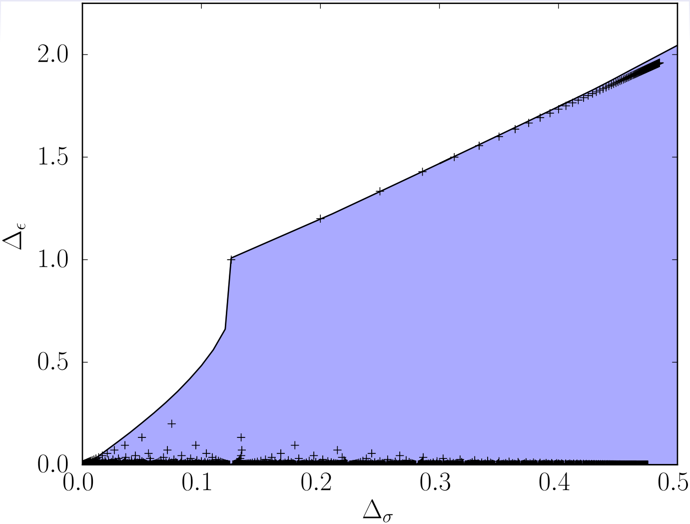


Set Δ_σ, Δ_ϵ to be various $\Delta_{r,s} = \frac{[r(m+1)-ms]^2-1}{2m(m+1)}$ for $m = 3, 4, \ldots$



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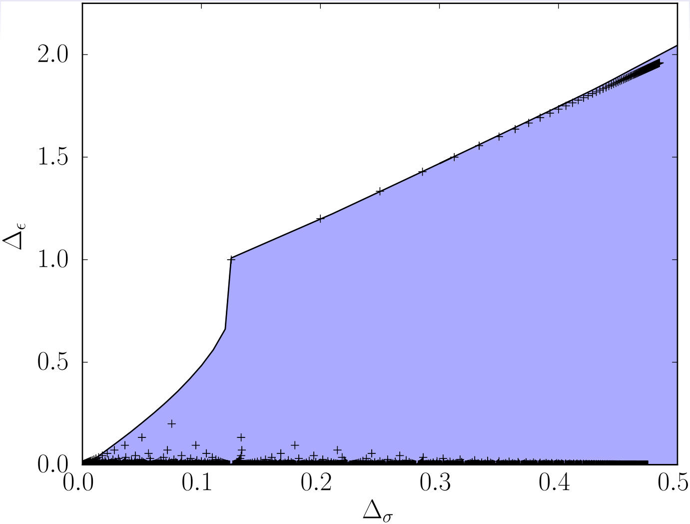

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Allowed points can be deformed by $\sigma\chi$ for any $\frac{d-2}{2} < \Delta_\chi < d - \Delta_\sigma$ [CB, Rastelli, Rychkov, Zan; 1703.05325] .

Non-unitarity is not always fatal

Along the upper bound, $\sigma = \phi_{(1,2)}$, $\epsilon = \phi_{(1,3)}$ leads to

$$\Delta_\epsilon = \frac{1}{3}(8\Delta_\sigma + 2)$$

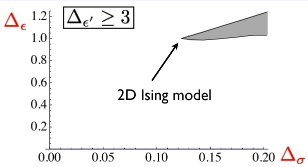
regardless of whether $m \in \mathbb{Z}$ or $m \in \mathbb{R}$. Null states in $V_{(r,s)}$ lead to

$$\phi_{(r_1,s_1)} \times \phi_{(r_2,s_2)} = \sum_{r_3=|r_1-r_2|+1}^{r_1+r_2-1} \sum_{s_3=|s_1-s_2|+1}^{s_1+s_2-1} \phi_{(r_3,s_3)} \Rightarrow$$

generalized minimal model [Zamolodchikov; 05] [Ribault; 1406.4290] . Decompose

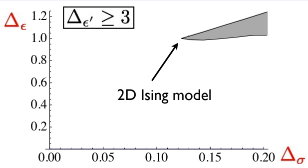
$$|z|^{2\Delta_\sigma} \left\langle \phi_{(1,2)}(0)\phi_{(1,2)}(z,\bar{z})\phi_{(1,2)}(1)\phi_{(1,2)}(\infty) \right\rangle = \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 G_{\Delta,\ell}(z,\bar{z})$$

and observe positivity [Liendo, Rastelli, van Rees; 1210.4258] .



Single correlator 2D Ising kink sharpens if next even scalar starts at $\Delta = 3$

[Rychkov; 1111.2115] .



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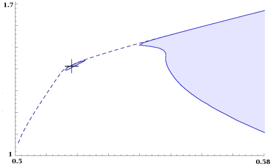
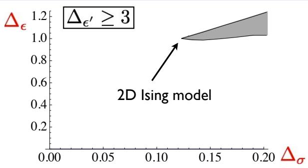


Ising island from $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\epsilon\epsilon\epsilon\epsilon\rangle$, $\langle\sigma\sigma\epsilon\epsilon\rangle$ system in 3D [Kos, Poland, Simmons-Duffin; 1406.4858] .
What about $\epsilon \not\subset \epsilon \times \epsilon$?



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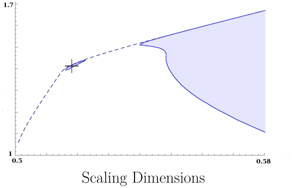

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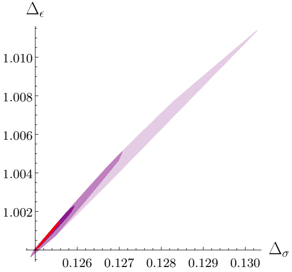


Repeating this for 2D Ising seems to require a spin-2 gap of ≈ 1 above $T_{\mu\nu}$ [de la Fuente; 1905.09801] .

Expanding the higher correlators

For $3 < m < 4$, negative coefficients appear in a single OPE.

$$\phi_{(1,3)} \times \phi_{(1,3)} = \phi_{(1,1)} + \phi_{(1,3)} + \phi_{(1,5)}.$$

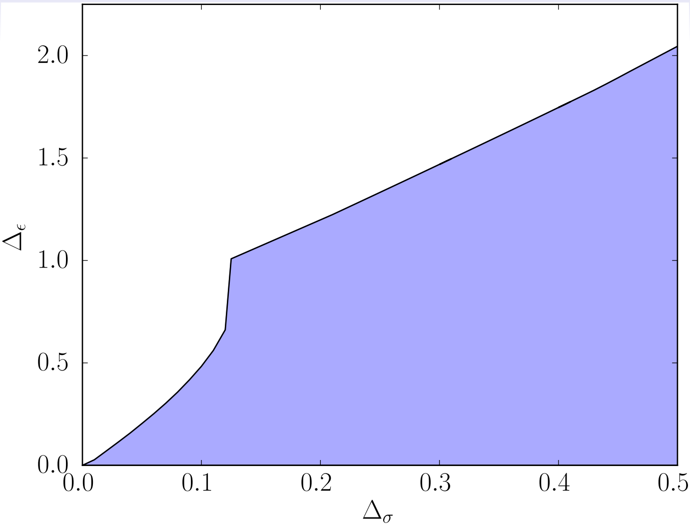
Finding operators to replace $\phi_{(1,5)}$ is a single correlator problem.

$$\sum_{\mathcal{O}} \lambda^2_{\epsilon\epsilon\mathcal{O}} \mathbf{F}^{\epsilon\epsilon;\epsilon\epsilon}_{\Delta,\ell} = -\mathbf{T}$$

$$\mathbf{T} = \sum_{n,\bar{n}=0}^{\infty} c_n^{(1,1)} c_{\bar{n}}^{(1,1)} \mathbf{F}^{\epsilon\epsilon;\epsilon\epsilon}_{n+\bar{n},|n-\bar{n}|}$$

$$+ C^{(1,3)}_{(1,3)(1,3)} \sum_{n,\bar{n}=0}^{\infty} c_n^{(1,3)} c_{\bar{n}}^{(1,3)} \mathbf{F}^{\epsilon\epsilon;\epsilon\epsilon}_{\Delta_\epsilon+n+\bar{n},|n-\bar{n}|}$$

Interestingly $C^{(1,5)}_{(1,3)(1,3)} < 0$ even at $m = 3 \Rightarrow$ Log-CFT?



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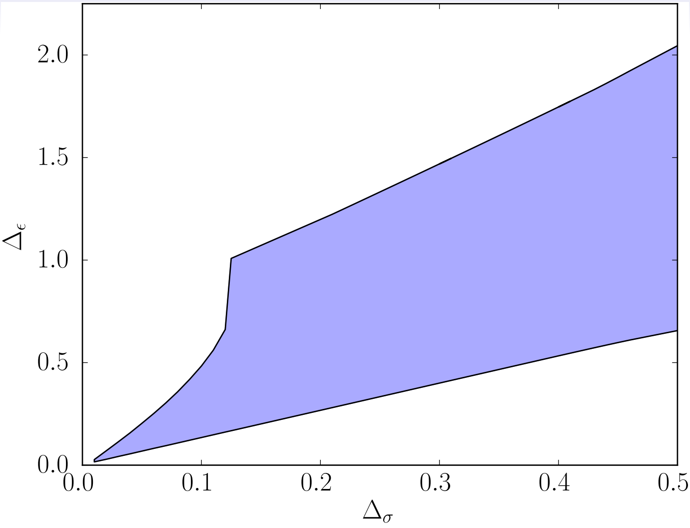

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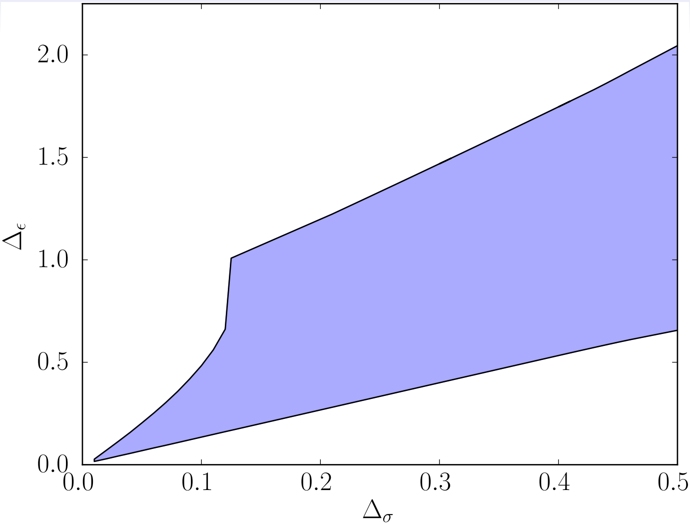
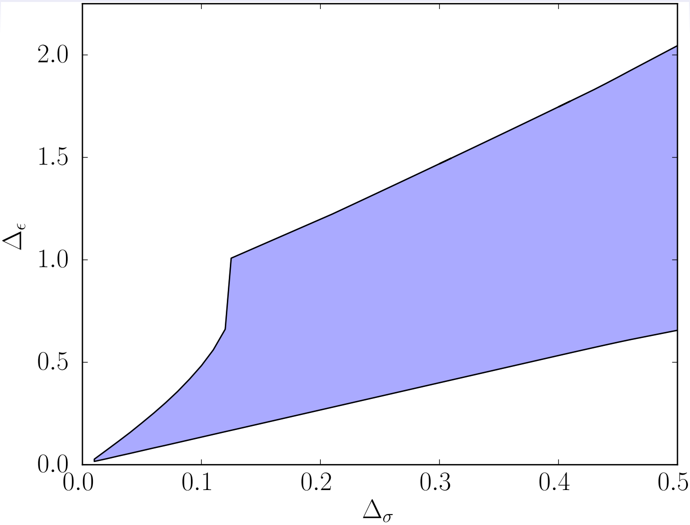


Non-unitary CFT on lower bound with $\Delta_\epsilon = \frac{4}{3}\Delta_\sigma$ was constructed in [Esterlis, Fitzpatrick, Ramirez; 1606.07458] .



Non-unitary CFT on lower bound with $\Delta_\epsilon = \frac{4}{3}\Delta_\sigma$ was constructed in [Esterlis, Fitzpatrick, Ramirez; 1606.07458] . Difficulty of solving for c_n, $c_{\bar{n}}$ depends on the correlator.

The lower bound

Correlator is a single Virasoro block.

$$|z|^{2\Delta_\sigma} \langle \sigma(0)\sigma(z,\bar{z})\sigma(1)\sigma(\infty)\rangle = g(z)g(\bar{z})$$
$$g(z) = z^{\frac{2}{3}\Delta_\sigma}(1-z)^{-\frac{1}{3}\Delta_\sigma}$$

Consider $V_\alpha = e^{i\alpha\phi}$ with background charge of $2\alpha_0$.

$$T(z) \mapsto T(z) + i\alpha_0 \partial^2 \phi(z)$$
$$c \mapsto 1 - 24\alpha_0^2$$
$$h_\alpha \mapsto \alpha(\alpha - 2\alpha_0)$$

If $\sigma = V_{\frac{\alpha_0}{2}} \bar{V}_{\frac{\alpha_0}{2}}$, we can skip inserting

$$Q_\pm \sim \oint dz V_{\alpha_0 \pm \sqrt{\alpha_0^2+1}}(z).$$

Ways to compute the expansion

1. Brute force:

$$g(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{3} \Delta_\sigma \right)_n z^{\frac{2}{3}\Delta_\sigma + n} = \sum_{n=0}^{\infty} c_n g^{0,0}_{\frac{2}{3}\Delta_\sigma + n}(z)$$

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2. Lorentzian inversion [Caron-Huot; 1703.00278] :

$$dDisc[g(z)g(\bar{z})] = 2\sin^2\left(\frac{\pi\Delta_\sigma}{3}\right) g(z)g(\bar{z})$$

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3. Compute spectral density for each half [Hogervorst, van Rees; 1702.08471] :

$$\hat{g}(\alpha) = \int_0^1 \frac{dz}{z^2} g(z) {}_2F_1\left(\frac{1}{2} + \alpha, \frac{1}{2} - \alpha; 1; \frac{z-1}{z}\right)$$

Shows that c_n is a ${}_3F_2$ function.

Proving positivity

Coefficients in

$$\frac{z^p}{(1-z)^q} = \sum_{n=0}^{\infty} \frac{(p)_n^2 n!^{-1}}{(2p+n-1)_n} {}_3F_2 \left[\begin{array}{c} -n, 2p+n-1, p-q \\ p, p \end{array} ; 1 \right] g_{p+n}^{0,0}(z)$$

are continuous Hahn polynomials satisfying a 3-term recursion.

$$P_{2n}\left(-\frac{1}{3}\Delta_\sigma\right) = \frac{3(2n-1)(\Delta_\sigma + 3n - 3)}{(2\Delta_\sigma + 3n - 3)(2\Delta_\sigma + 6n - 3)} P_{2n-2}\left(-\frac{1}{3}\Delta_\sigma\right)$$

Could also use Watson's theorem when $p = 2q$.

$${}_3F_2 \left[\begin{array}{c} a, b, c \\ \frac{a+b+1}{2}, 2c \end{array} ; 1 \right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1+a+b}{2})\Gamma(\frac{1}{2}+c)\Gamma(\frac{1-a-b}{2}+c)}{\Gamma(\frac{1+a}{2})\Gamma(\frac{1+b}{2})\Gamma(\frac{1-a}{2}+c)\Gamma(\frac{1-b}{2}+c)}$$

Generalized minimal models

For all m, $\left\langle\phi_{(r,s)}\dots\right\rangle$ obeys a PDE in the cross-ratios of order rs.
Series around $z=0$ fixed by Frobenius method with known h.

$$G(z) = z^h\left[1 + b_1 z + b_2 z^2 + \dots\right].$$

Take diagonal linear combination with [Dotsenko, Fateev; 85] coefficients.

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In second order case,

$$G^{\sigma\sigma\sigma\sigma}_{(1,1)}(z) = (1-z)^{-\Delta_\sigma} {}_2F_1\left(-2\Delta_\sigma, \frac{1-2\Delta_\sigma}{3}; \frac{2-4\Delta_\sigma}{3}; z\right)$$

$$G^{\sigma\sigma\sigma\sigma}_{(1,3)}(z) = z^{\frac{1+4\Delta_\sigma}{3}} (1-z)^{-\Delta_\sigma} {}_2F_1\left(\frac{1-2\Delta_\sigma}{3}, \frac{2+2\Delta_\sigma}{3}; \frac{4+4\Delta_\sigma}{3}; z\right)$$

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yielding c_n as a hard sum of ${}_3F_2$s. Instead use

$$G^{\sigma\sigma\sigma\sigma}_{(1,1)}(z) = {}_2F_1\left(-\Delta_\sigma, \frac{1+\Delta_\sigma}{3}; \frac{5-4\Delta_\sigma}{6}; \frac{1}{4}\frac{z^2}{z-1} \right)$$

$$G^{\sigma\sigma\sigma\sigma}_{(1,3)}(z) = z^{\frac{1+4\Delta_\sigma}{3}} (1-z)^{-\frac{1+4\Delta_\sigma}{6}} {}_2F_1\left(\frac{1+2\Delta_\sigma}{2}, \frac{1-2\Delta_\sigma}{6}; \frac{7+4\Delta_\sigma}{6}; \frac{1}{4}\frac{z^2}{z-1} \right)$$

yielding c_n as an easy sum of ${}_3F_2$s.

Results

Coefficients are Wilson polynomials

$$c_{2n}^{(1,1)} = \binom{4n-2}{2n-1}^{-1} \frac{\Delta_\sigma(1+\Delta_\sigma)}{5-4\Delta_\sigma} {}_4F_3\left[\begin{matrix} 1-n, n+\frac{1}{2}, 1-\Delta_\sigma, \frac{4+\Delta_\sigma}{3} \\ 1, 2, \frac{11-4\Delta_\sigma}{6} \end{matrix} ; 1\right]$$

$$c_{2n}^{(1,3)} = \frac{4^{-n} n!^{-1} \left(\frac{1+4\Delta_\sigma}{6}\right)_n^2}{\left(\frac{1+4\Delta_\sigma}{3}+n-\frac{1}{2}\right)_n} {}_4F_3\left[\begin{matrix} -n, \frac{-1+8\Delta_\sigma}{6}+n, \Delta_\sigma+\frac{1}{2}, \frac{1-2\Delta_\sigma}{6} \\ \frac{1+4\Delta_\sigma}{6}, \frac{1+4\Delta_\sigma}{6}, \frac{7+4\Delta_\sigma}{6} \end{matrix} ; 1\right]$$

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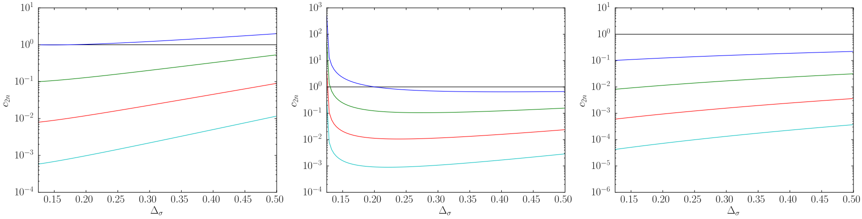
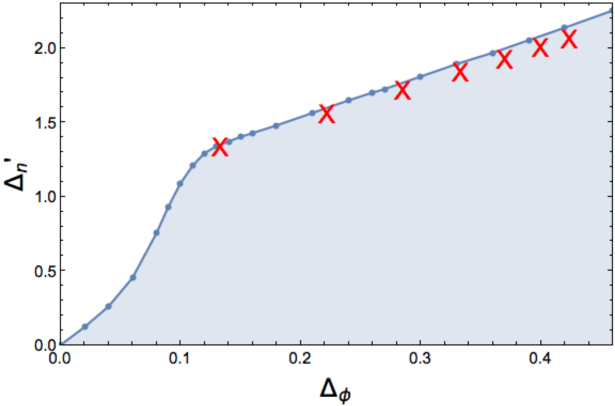
obeying a 3-term recursion $\Rightarrow$ positivity proof.

Solution for $\langle \epsilon\epsilon\epsilon\epsilon \rangle$ still relies on brute force.



Next target



2D S_3 bootstrap knows about W_3 minimal models [Rong, Su; 1712.00985] .

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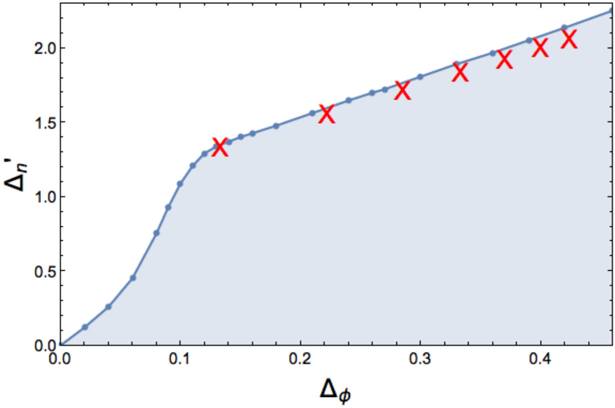

2D S_3 bootstrap knows about W_3 minimal models [Rong, Su; 1712.00985] .
Null state conditions reviewed in [Fateev, Litvinov; 0709.3806] .

A better Frobenius method

The key identity is

$$f_{p,q}(z) = z^{h-p} {}_2F_1\left(\begin{matrix} h, h-q \\ 2h \end{matrix} ; z \right)$$

$$\frac{d}{dz} f_{p,q}(z) = (h-p) f_{p+1,q}(z) + \frac{1}{1-z}\left[(h+q) f_{p+1,q+1}(z) - (h+q) f_{p+1,q}(z) + h f_{p,q}(z)\right].$$

Differentiated block $f_{p,q}$ expands into finitely many blocks.

$$c_n = A(n)c_{n-1} + B(n)c_{n-2} + C(n)c_{n-3} \Rightarrow$$
$$c_n \geq \tilde{A}(n)c_{n-1} + \tilde{B}(n)c_{n-2}?$$

A single recursion might enable more positivity proofs.