# Fields all the way down

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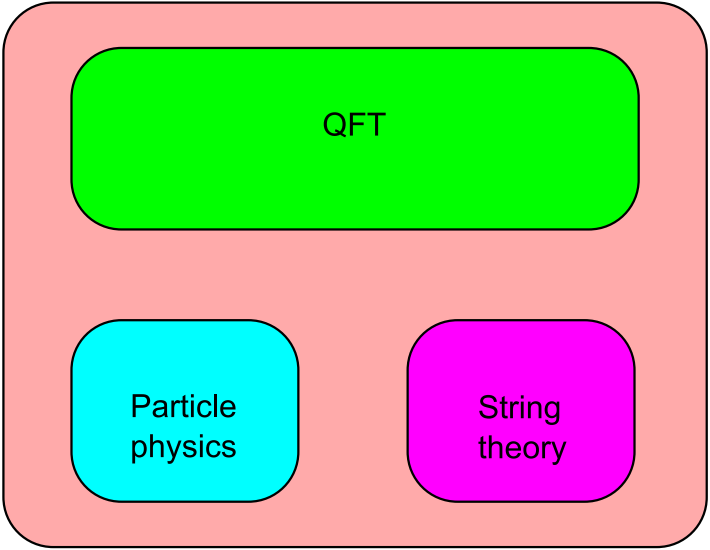
# Specializations of quantum mechanics



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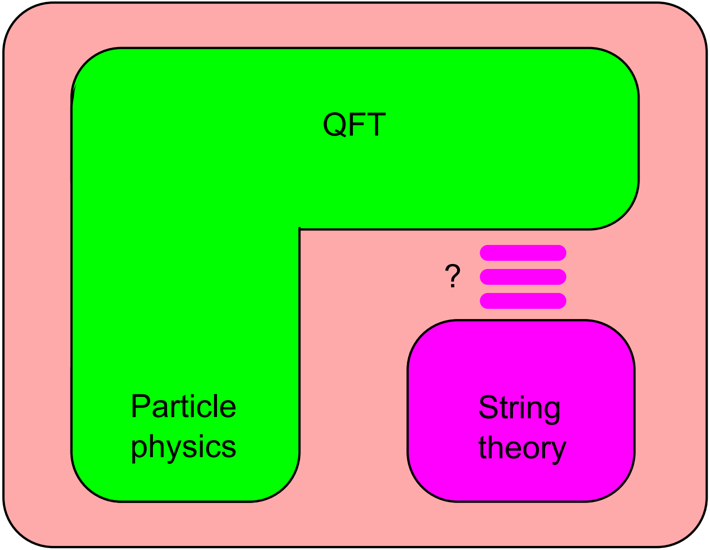
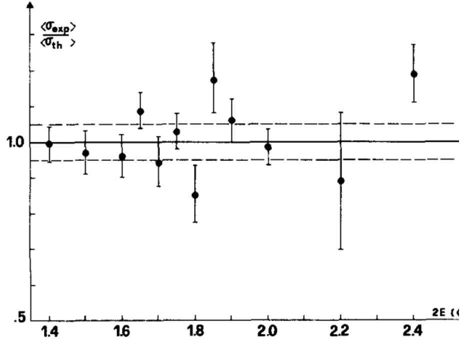
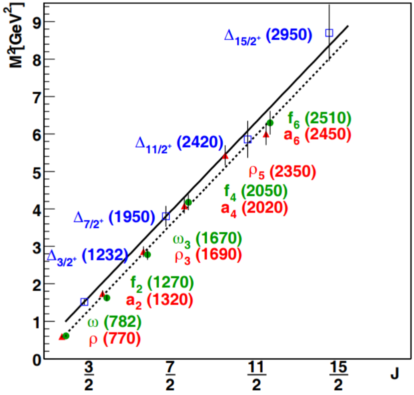


# Worldlines and worldsheets

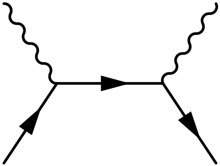



[Borgia + 9; 1971]




[Klemt, Metsch; 2012]

# Scattering processes



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$$H = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m_i} + \sum_{i<j} V(x_i, x_j)$$

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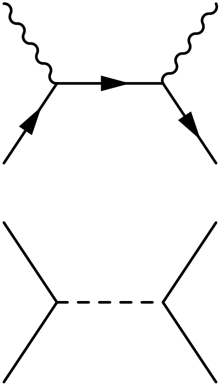

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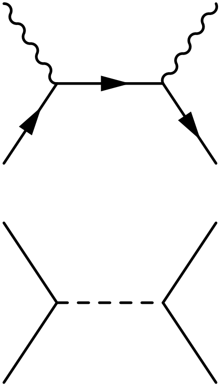

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Can build amplitude out of [Feynman; 1950]

$$D_{yz} = \int_0^\infty dT \int_{x(0)=y}^{x(T)=z} Dx \, e^{-S} = \int_0^\infty dT (4\pi T)^{-\frac{d}{2}} e^{-m_0^2 T - \frac{(y-z)^2}{4T}}$$

# Working with fields

A field theory for this is

$$S = \int dx \, \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m_0^2 \phi^2 + \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2} M_0^2 \Phi^2 + g_0 \phi^2 \Phi$$

Free EOM is $\partial^2 \phi + m_0^2 \phi = 0$ and same for $\Phi$ so

$$\phi(x,t) = \sum_{\mathbf{p}} C_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x} - \sqrt{\mathbf{p}^2 + m_0^2} t)} + C^*_{\mathbf{p}} e^{-i(\mathbf{p}\cdot\mathbf{x} - \sqrt{\mathbf{p}^2 + m_0^2} t)}$$

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Now evolve with $U = T \exp\left(-i \int_{-\infty}^{\infty} dt\, H_{int}\right)$ to get

$$\begin{aligned}
A &= \langle 0 |\, a_{p_4} a_{p_3} \exp\left(g_0 \int dx\, \phi^2 \Phi\right) a_{p_2}^\dagger a_{p_1}^\dagger \,| 0 \rangle \\
&\approx g_0^2 \int dxdy\, \langle 0 |\, a_{p_4} a_{p_3} \phi^2 \Phi(x) \phi^2 \Phi(y) a_{p_2}^\dagger a_{p_1}^\dagger \,| 0 \rangle \\
&= g_0^2 \int dxdy\, \langle a_{p_4} \phi(x) \rangle \langle a_{p_3} \phi(x) \rangle \langle \Phi(x) \Phi(y) \rangle \left\langle \phi(y) a_{p_2}^\dagger \right\rangle \left\langle \phi(y) a_{p_1}^\dagger \right\rangle \\
&= g_0^2 \int dp\, e^{ip\cdot(x-y)} \frac{1}{p^2 + M_0^2}
\end{aligned}$$

# Massless particles with spin

As a bispinor $p_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$ so use

$$\langle ij \rangle = \lambda_{i\alpha} \lambda_{j\beta} \epsilon^{\alpha\beta}, \quad [ij] = \bar{\lambda}_{i\dot{\alpha}} \bar{\lambda}_{j\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}}$$

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Use dimensional analysis, $\lambda \to t\lambda$, $\bar{\lambda} \to t^{-1}\bar{\lambda}$ and

$$\varepsilon^{+}_{\alpha\dot{\alpha}} = \frac{\eta_\alpha \bar{\lambda}_{\dot{\alpha}}}{\langle \eta\lambda \rangle}, \quad \varepsilon^{-}_{\alpha\dot{\alpha}} = \frac{\lambda_\alpha \bar{\eta}_{\dot{\alpha}}}{[\bar{\lambda}\bar{\eta}]}$$

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to bootstrap amplitudes. 3 points, have only $\langle ij \rangle$ or only $[ij]$.

$$A(1^{-h}, 2^{-h}, 3^{+h}) = \frac{\langle 12 \rangle^{3h}}{\langle 23 \rangle^h \langle 31 \rangle^h}, \quad A(1^{+h}, 2^{+h}, 3^{-h}) = \frac{[12]^{3h}}{[23]^h [31]^h}$$

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With 4 points, factorization $\lim_{s\to 0} sA_4 = A_3 A_3$ restricts helicity!

# Allowed spacetime symmetries

QFT in $d > 2$ with scattering has only Poincaré [Coleman, Mandula; 1967]

$$\begin{aligned} p_1^\mu p_1^\nu + p_2^\mu p_2^\nu &\propto \langle p_1 | Q^{\mu\nu} | p_1 \rangle + \langle p_2 | Q^{\mu\nu} | p_2 \rangle \\ &= \langle p_1, p_2 | Q^{\mu\nu} | p_1, p_2 \rangle \\ &= \langle q_1, q_2 | Q^{\mu\nu} | q_1, q_2 \rangle \\ &= \langle q_1 | Q^{\mu\nu} | q_1 \rangle + \langle q_2 | Q^{\mu\nu} | q_2 \rangle \propto q_1^\mu q_1^\nu + q_2^\mu q_2^\nu \end{aligned}$$

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&6D \Rightarrow \mathfrak{osp}(8^*|\mathcal{N}), \quad \mathcal{N} = 1, 2 \\
&5D \Rightarrow \mathfrak{f}(4) \\
&4D \Rightarrow \mathfrak{su}(2,2|\mathcal{N}), \quad \mathcal{N} = 1, \dots, 4 \\
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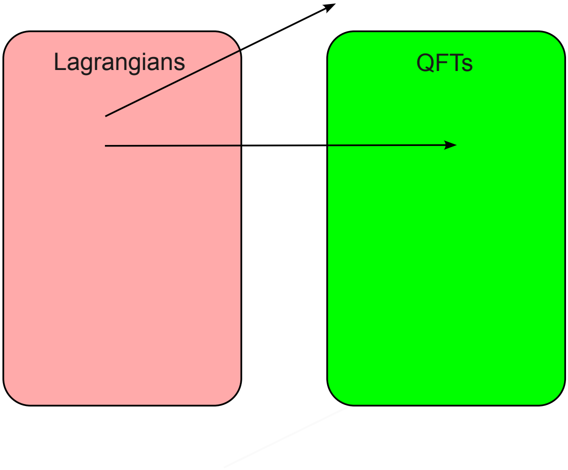
Many charges in 2D which generate $z \mapsto z + \epsilon z^{n+1}$:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} n(n^2 - 1)\delta_{m+n,0}$$

# A pernicious misconception



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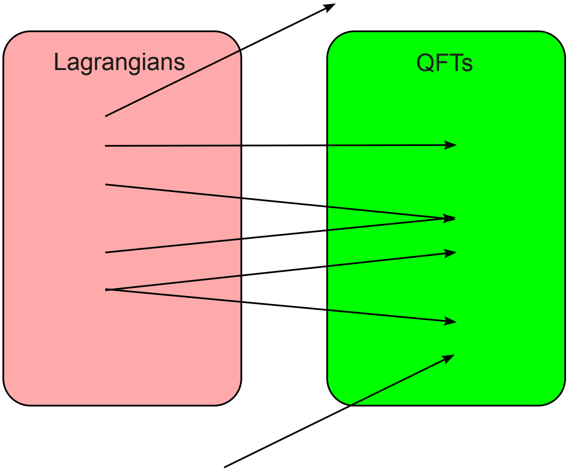
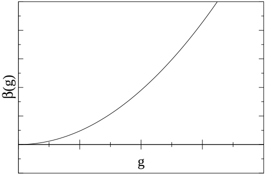


# Ways to get a conformal theory

$$S = \int d^4x \, \frac{1}{2}(\partial\phi)^2 + \frac{g_0}{4!}\phi^4$$

$$\beta(g) \equiv \frac{dg}{d\mu} = \frac{3g^2}{(4\pi)^2}$$




- One-loop diagram $F$ is a divergent function of $g_0$.
- In turn, make $g_0$ a divergent function of $g$ to cancel infinities.
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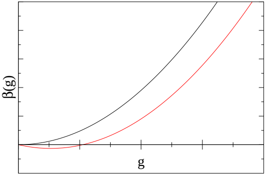


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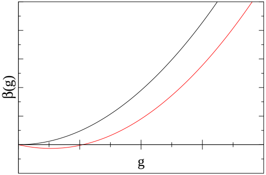


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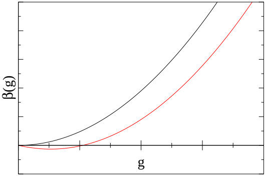


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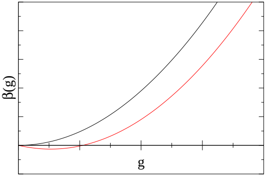


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$$G_{\mu\nu}(X) = \eta_{\mu\nu} + \partial_\rho G_{\mu\nu}(x_0)\delta X^\rho + \partial_\rho\partial_\sigma G_{\mu\nu}(x_0)\delta X^\rho \delta X^\sigma + \ldots$$

Vanishing beta gives relations like $R_{\mu\nu} = 0$.

# Scattering strings instead of particles

Path integrate over manifolds, not just metrics:

$$A = \sum_{top} \int DXDg \, e^{-S-\lambda S_{top}} \ldots$$

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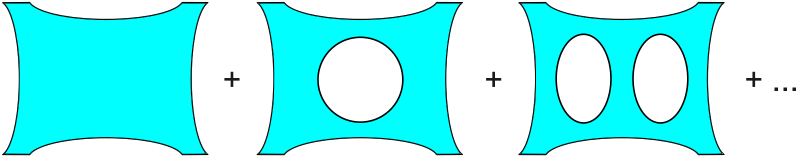

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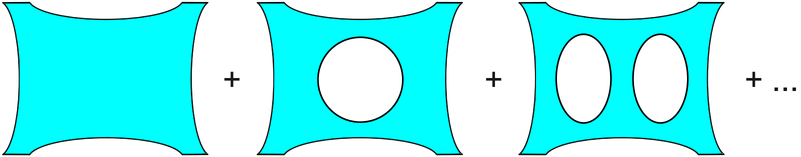


Many fermions with $\frac{\lambda}{N}(\bar{\psi}\psi)^2$ reveals surprise [Gross, Neveu; 1974] :

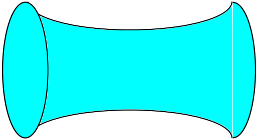
$$S = \int d^2z \, \bar{\psi}\gamma^\mu \partial_\mu \psi - \frac{N}{2\lambda}\sigma^2 + \sigma\bar{\psi}\psi \to N Tr \log(1 - \sigma\partial^{-2}\sigma) + \int d^2z \, \frac{N}{\lambda}\sigma^2$$

Large $N$ YM also gives hints that string theory is a QFT ['t Hooft; 1974] .

# Anti-de Sitter space

Isometry group $SO(d+1,1)$ is also conformal group.

$AdS_{d+1}$:
$-X_0^2 + X_1^2 + \dots X_{d+1}^2 = -L^2$



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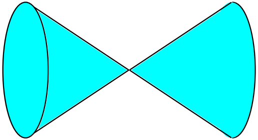

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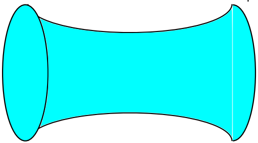

Poincaré $(X_0, X_\mu, X_{d+1}) =$

$L\left(\frac{1+x^2+z^2}{2z}, \frac{x_\mu}{z}, \frac{1-x^2-z^2}{2z}\right)$

$ds^2 = \frac{L^2}{z^2}\left(dz^2 + dx_\mu dx^\mu\right)$

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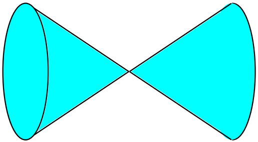

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$\mathcal{O}(P) = \lambda^\Delta \mathcal{O}(P/\lambda)$

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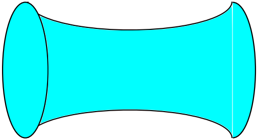

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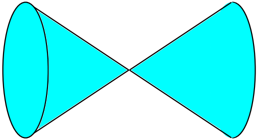
$CFT_d$:

$-P_0^2 + P_1^2 + \cdots + P_{d+1}^2 = 0$

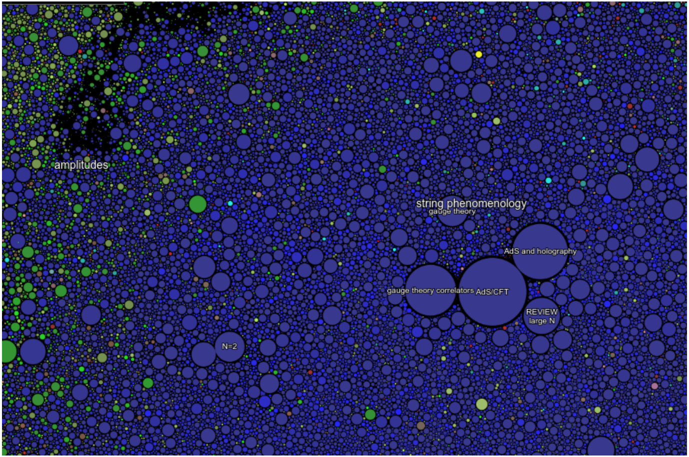


Poincaré $(P_0, P_\mu, P_{d+1}) =$

$\left(\frac{1+x^2}{2}, x_\mu, \frac{1-x^2}{2}\right)$

$\mathcal{O}(P) = \lambda^\Delta \mathcal{O}(P/\lambda)$

- Casimir with eigenvalue $\Delta(\Delta - d)$ is $(P_A\partial_B - P_B\partial_A)^2$.
- Laplacian in $(\partial^2 + m^2)\Phi = 0$ is $(X_A\partial_B - X_B\partial_A)^2$.
- Green's function with $X$ and $P$ points gives bulk-bulk and bulk-boundary propagators leading to CFT correlators.




Have fun exploring!